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**Mathematical Publications**

# DENJOY AND $\mathcal{P}$-PATH INTEGRALS ON COMPACT GROUPS IN AN INVERSION FORMULA FOR MULTIPLICATIVE TRANSFORMS

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ABSTRACT. Denjoy and $\mathcal{P}$-path Kurzweil-Henstock type integrals are defined on compact subsets of some locally compact zero-dimensional abelian groups. Those integrals are applied to obtain an inversion formula for the multiplicative integral transform.

## 1. Introduction

In [8], an inversion formula for the multiplicative integral transform was obtained for the case of any locally compact zero-dimensional abelian periodic group. This result does not cover the case of transforms convergent to Denjoy-Khintchine integrable functions because this integral is incompatible with $\mathcal{P}$-adic integral used in the above result (see [2]).

To overcome this difficulty, we consider here a little bit less general class of zero-dimensional groups and another Kurzweil-Henstock type integral on them. This gives us an opportunity to get an inversion formula for the multiplicative integral transform convergent to a Denjoy-Khintchine integrable function in the case of a class of groups considered here.

This problem is a generalization of that of recovering the coefficients of a convergent series with respect to characters of a compact zero dimensional abelian group which was considered in [7]. A similar problem related to some special groups was considered in [6].

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## 2. Preliminaries

Let $G$ be a zero-dimensional locally compact abelian periodic group which satisfies the second countability axiom. We can introduce (see [1]) a topology in such a group using a chain of subgroups

$$\ldots \supset G_{-n} \supset \ldots \supset G_{-2} \supset G_{-1} \supset G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_n \supset \ldots$$

with $G = \cup_{n=-\infty}^{+\infty} G_n$ and $\{0\} = \cap_{n=-\infty}^{+\infty} G_n$. The subgroups $G_n$ are clopen sets with respect to this topology. As $G$ is periodic, the factor group $G_n/G_{n+1}$ is finite for each $n$ and this implies that $G_n$ is compact. Let $p_{n+1}$ be an order of $G_n/G_{n+1}$ if $n \geq 0$ and $p_{-n}$ an order of $G_{-n}/G_{-n+1}$ if $n \geq 1$. We can suppose that all $p_n$ are prime numbers. So, the group $G$ defines a sequence

$$\mathcal{P} = \{p_j\}_{j=-\infty, \qquad j\neq 0}^{+\infty} . \tag{1}$$

We also consider the reverse sequence

$$\mathcal{P}' = \{p'_j\}_{j=-\infty, \qquad j\neq 0}^{+\infty} , \tag{2}$$

where $p'_j = p_{-j}$ for $j \in \mathbb{Z}\backslash\{0\}$. Moreover, we set $m_0 = 1$, $m_j = \prod_{s=1}^{j} p_s$, $m_{-j} = \prod_{s=-1}^{-j} p_s$.

We remind that a complex function $\chi$ on a locally compact abelian group $G$ is called a *character* of $G$ if $|\chi(g)| = 1$ for all $x \in G$ and if a functional equation

$$\chi(g_1 + g_2) = \chi(g_1) + \chi(g_2)$$

is satisfied for all $g_1, g_2 \in G$. The set of all continuous characters of $G$ forms a group $X$, the *dual group* of $G$, if the addition is defined by

$$(\chi_1 + \chi_2)(g) = \chi_1(g) \cdot \chi_2(g),$$

where $x \in G$ and $\chi_1, \chi_2 \in X$. In view of the duality between $G$ and $X$, see ([10]), it is customary to write $(g, \chi)$ in place of $\chi(g)$.

In what follows, $X$ will denote the dual group of the group $G$ described above. It is known (see [1]) that under the assumption imposed on $G$ the group $X$ is also a periodic locally compact zero-dimensional abelian group and we can represent it to be a sum of increasing sequence of subgroups:

$$\ldots \supset X_n \supset \ldots \supset X_2 \supset X_1 \supset X_0 \supset X_{-1} \supset X_{-2} \supset \ldots \supset X_{-n} \supset \ldots \tag{3}$$

introducing a topology in $X$. Then, $X = \cup_{i=-\infty}^{+\infty} X_i$ and $\cap_{i=-\infty}^{+\infty} X_i = \{\chi^{(0)}\}$ where $\big(g, \chi^{(0)}\big) = 1$ for all $g \in G$. For each $n \in \mathbb{Z}$, the group $X_n$ is the annihilator of $G_n$, i.e.,

$$X_n = G_n^{\perp} = \big\{\chi \in X : (g, \chi) = 1 \qquad \text{for all} \quad g \in G_n\big\}.$$

The factor groups $X_{n+1}/X_n = G_{n+1}^{\perp}/G_n^{\perp}$ and $G_n/G_{n+1}$ are isomorphic (see [1]). So, the order of $X_{n+1}/X_n$ is $p_{n+1}$ if $n \geq 0$ and $p_n$ if $n \leq -1$.

It can be shown (see [8]) that any element $g \in G$ can be represented in the form

$$g = [g] + \{g\} \tag{4}$$

and any element $\chi \in X$ in the form

$$\chi = [\chi] \cdot \{\chi\} \tag{5}$$

in such a way that the following properties hold:

1) $\{g\} \in G_0$ and $\{\chi\} \in X_0 = G_0^{\perp}$. So, $\big(\{g\}, \{\chi\}\big) = 1$, and
$$(g, \chi) = \big(\{g\}, [\chi]\big) \cdot \big([g], [\chi]\big) \cdot \big([g], \{\chi\}\big). \tag{6}$$
2) $[\chi] \in G_m^{\perp}$ for some $m = m(\chi) \in I\!N$ and $[\chi]\backslash_{G_0}$ is a character of the subgroup $G_0$.
3) $\big([g], [\chi]\big)$ is constant if $g$ belongs to a fixed coset of $G_0$ and $\chi$ belongs to a fixed coset of $X_0$.

Using the duality between $G$ and $X$, we can state that $g$ represents a character of $X$ and similarly to property 2), $[g]\backslash_{X_0}$ is a character of $X_0$. So, $\big([g], \{\chi\}\big)$ is a value of this character at the point $\{\chi\}$.

Therefore, according to 6), if $g$ belongs to a fixed coset of $G_0$ and $\chi$ belongs to a fixed coset of $X_0$, we can represent $(g, \chi)$, up to a constant $\big([g], [\chi]\big)$, as a product of $\big(\{g\}, [\chi]\big)$ considered as a value of the character $[\chi]$ at $\{g\}$, and $\big([g], \{\chi\}\big)$ considered as a value of the character $[g]$ at $\{\chi\}$.

It is also shown (see [8]) that we can map the groups $G$ and $X$ to the interval $[0, +\infty)$ by mappings

$$\varphi(g) = x = \{x\} + [x] \tag{7}$$

and

$$\psi(\chi) = x' = \{x'\} + [x'], \tag{8}$$

where $\{a\}$ is the fractional part of $a$ and $[a]$ the integer part of $a$. Moreover, the preimage of $[x]$, under the mapping $\varphi$, is $[g]$ and the one of $\{x\}$ is $\{g\}$. This explains the notation in (4). In the same way, $\psi\big([\chi]\big) = [x']$ and $\psi\big(\{\chi\}\big) = \{x'\}$.

In this way, the subgroup $G_n, n \geq 0$, is mapped onto the interval $\big[0, \frac{1}{m_n}\big]$ and the respective cosets are mapped onto intervals $I_n^k = \big[\frac{k}{m_n}, \frac{k+1}{m_n}\big]$ with $k = 0, 1, \ldots$ We denote these cosets by $G_n^k$ with $G_n^0 = G_n$. In particular, for $n = 0$ we map $G_0$ onto $[0, 1]$ and respective cosets onto intervals $[k, k+1]$. In case $n < 0$, the group $G_n$ is mapped onto interval $[0, m_n]$ and respective cosets $G_n^k$ are mapped onto intervals $[km_n, (k+1)m_n]$. We call all the above mentioned images of cosets the *$\mathcal{P}$-adic intervals*. If $n$ is fixed, we refer to $I_n^k$ as to intervals of a rank $n$. The set of all $\mathcal{P}$-adic intervals is denoted by $\mathcal{I}_{\mathcal{P}}$.

In a similar way, we can define $\mathcal{P}'$-adic intervals associated with the group $X$ and the sequence $\mathcal{P}'$ (see (2)). They are images, under mapping (8), of the respective cosets of the subgroup $X_n$ denoted by $X_n^s$.

Note that the mappings (7) and (8) are one-one correspondences between the interval $[0, +\infty)$ and the groups $G$ and $X$, respectively, up to a countable number of $\mathcal{P}$-adic ($\mathcal{P}'$-adic) rational points, i.e., points $\frac{t}{m_k}$ (or $\frac{t}{m_{-k}}$) with $t, k = 0, 1, 2, \ldots$ Such a point $x$ has two preimages corresponding to the finite and to the infinite expansion of $x$, respectively. We denote by $Q_{\mathcal{P}}$ the set of all $\mathcal{P}$-adic rational points and by $Q_{\mathcal{P}'}$ the set of all $\mathcal{P}'$-adic rational points. We agree to use only finite expansions for $\mathcal{P}$-adic (and $\mathcal{P}'$-adic) rational points so that the inverse mappings $\varphi^{-1}$ and $\psi^{-1}$ make sense on $[0, \infty)$.

We consider the Haar measure $\mu_G$ on the group $G$ and we normalize it so that $\mu_G(G_0) = 1$. Then the measure of any coset $G_n^k$ of $G_n$ coincides with the length of the $\mathcal{P}$-adic interval which is the image of $G_n^k$, i.e.,

$$\mu_G(G_n^k) = \frac{1}{m_n} \quad \text{if} \quad n \geq 0 \qquad \text{and} \qquad \mu_G(G_n^k) = m_n \quad \text{if} \quad n < 0.$$

In the same way, we introduce the measure $\mu_X$ on $X$ so that $\mu_X(X_0) = 1$ and

$$\mu_X(X_n^s) = \frac{1}{m_n} \quad \text{if} \quad n \leq 0 \qquad \text{and} \qquad \mu_X(X_n^s) = m_n \quad \text{if} \quad n > 0.$$

Note that, under the above mentioned mappings $\varphi$ and $\psi$, the image of each set of Haar measure zero on the group is a set of Lebesgue measure zero on $[0, +\infty)$.

We denote by $\mathcal{R}_G$ the ring generated by the family of all cosets $G_n^k$, $n \in \mathbb{Z}$, $k = 0, 1, 2, \ldots$, and by $\mathcal{R}_{\mathcal{P}}$ the ring generated by $\mathcal{I}_{\mathcal{P}}$. Note that for each $g \in G$ there exists a decreasing sequence $\{G_n^{k(n)}\}_n$ of cosets such that $g \in \cap_n G_n^{k(n)}$.

## 3. $\mathcal{P}$-path Kurzweil-Henstock integral and Denjoy-Khintchine integral on the group G

Now, we introduce a Henstock-Kurzweil type integral with respect to the system of $\mathcal{P}$-paths and also Denjoy-Khintchine integral on compact subgroups of the considered group $G$ and $X$.

With each $\mathcal{P}$-adic irrational point $x \in [0, \infty)$, i.e., a point $x \in [0, \infty) \setminus Q_{\mathcal{P}}$, we associate a unique nested sequence

$$\Big\{\varphi\Big(G_n^{k(n)}\Big)\Big\}_n = \Big\{I_n^{k(n)}\Big\} = \Big\{\big[a_n(x), b_n(x)\big]\Big\}_{n=0}^{\infty} \tag{9}$$

of $\mathcal{P}$-adic intervals converging to $x$ so that

$$\{x\} = \bigcap_{n=0}^{\infty} \big[a_n(x), b_n(x)\big].$$

If $x$ is a $\mathcal{P}$-adic rational point, then there exist two nested sequences (9) — the *left* one and the *right* one — for which $x$, starting with some $n$, is the common end-point.

Using the notation given by (9), we define for a $\mathcal{P}$-adic irrational point $x$, the sequences $\mathcal{P}_x^- = \{a_n(x)\}$ and $\mathcal{P}_x^+ = \{b_n(x)\}$. If $x$ is a $\mathcal{P}$-adic rational point, we use the same notation $\mathcal{P}_x^+$ and $\mathcal{P}_x^-$ for sequences with $a_n(x)$ being the left end-point of interval of rank $n$ of the above mentioned left nested sequence associated with $x$, and $b_n(x)$ being the right end-point of interval of rank $n$ of the right nested sequence.

**Definition 3.1.** The set $\mathcal{P}_x = \mathcal{P}_x^+ \bigcup \mathcal{P}_x^- \bigcup \{x\}$ is called the *$\mathcal{P}$-path leading to $x$*. If $E \in \mathcal{R}_\mathcal{P}$, the collection $\{\mathcal{P}_x : x \in E\}$ is called the *system of $\mathcal{P}$-paths on $E$*.

The continuity and the derivative at a point $x$ with respect to the set $\mathcal{P}_x$ are called *$\mathcal{P}$-path continuity* and *$\mathcal{P}$-path derivative*. In the same way, we define *$\mathcal{P}$-path upper* and *lower derivatives*.

**Definition 3.2.** Let $\delta$ be a positive function defined on $E \in \mathcal{R}_\mathcal{P}$. A collection of interval-point pairs $\{([u_j, v_j], x_j)\}_{j=1}^k$ is called *$\delta$-fine $\mathcal{P}$-partition of $E$* if the intervals $[u_j, v_j]$ are non-overlapping, $E = \cup_{j=1}^k [u_j, v_j]$ and for each $j$ we have $u_j, v_j \in \mathcal{P}_x$, $u_j \le x_j \le v_j$ and $\max\{v_j - x_j, x_j - u_j\} < \delta(x_j)$.

It is not difficult to check that for any $E \in \mathcal{R}_\mathcal{P}$, any system of $\mathcal{P}$-paths on $E$, and for any positive function $\delta$ defined on $E$ there exists a $\delta$-fine $\mathcal{P}$-partition of $E$.

**Definition 3.3.** A complex valued function $f$ on $E \in \mathcal{R}_\mathcal{P}$ is said to be *Kurzweil-Henstock integrable with respect to the system of $\mathcal{P}$-paths* or, in brief, *$H_\mathcal{P}$-integrable* on $E$, with integral value $A$, if for every $\varepsilon > 0$, there exists a positive function $\delta$ on $E$ such that for any $\delta$-fine $\mathcal{P}$-partition of $E$ we have

$$\left| \sum_{j=1}^{k} f(x_j)(v_j - u_j) - A \right| < \varepsilon.$$

Then we write $(H_\mathcal{P}) \int_E f = A$.

It is easy to check that a function which is equal to zero almost everywhere on $E \in \mathcal{R}_\mathcal{P}$, is $H_\mathcal{P}$-integrable to zero on $E$. This justifies the following extension of Definition 3.3 to the case of functions defined only almost everywhere on $E$ (for short, a.e.).

**Definition 3.4.** A complex valued function $f$ defined a.e. on $E \in \mathcal{R}_\mathcal{P}$ is said to be *$H_\mathcal{P}$-integrable* on $E$ with integral value $A$ if the function

$$f_1(x) = \begin{cases} f(x), & \text{where } f \text{ is defined,} \\ 0, & \text{otherwise,} \end{cases}$$

is $H_{\mathcal{P}}$-integrable on $E$ to $A$ in the sense of Definition 3.3.

We can use this definition to introduce the respective $H_G$-integral on certain subsets of the considered group $G$.

**Definition 3.5.** Let $E \in \mathcal{R}_{\mathcal{P}}$ be the image of $T \in \mathcal{R}_G$ under the mapping $\varphi$ (see (7)). Then a complex-valued function $f$ defined almost everywhere on $T$ is *$H_G$-integrable on $T$,* with the value $A$, if the function $F(x) = f\big(\varphi^{-1}(x)\big)$ is $H_{\mathcal{P}}$-integrable on $E$ with the value $A$ in the sense of Definition 3.4, and we write $(H_G)\int_T f d\mu_G = A$.

**Remark 3.1.** We note that the above definition depends on the sequence $\mathcal{P}$ defined by the group $G$. So, if we consider a similar definition of the $H_X$-integral on a subset of the group $X$, then we should use a sequence $\mathcal{P}'$ defined by $X$ and the respective $H_{\mathcal{P}'}$-integral.

Now, we shall give a definition of *Denjoy-Khintchine integral* on compact subgroups of a locally compact group.

**Definition 3.6.** A real function $\Phi$ defined on $[a, b]$ is an *ACG-function* if $[a, b] = \cup_{i=1}^{\infty} B_i$ and $\Phi$ is absolutely continuous on each $B_i$.

**Definition 3.7.** A real function $f$ defined almost everywhere on $[a, b]$ is *Denjoy--Khintchine integrable on* $[a, b]$, briefly, *D-integrable*, if there exists an $ACG$ function $\Phi$ such that $\Phi'_{ap} = f$ almost everywhere, where $\Phi'_{ap}$ denotes the approximate derivative (see [4]). The value of this integral is defined as $(D)\int_a^b f = \Phi(b) - \Phi(a)$.

**Definition 3.8.** A real function $f$ defined almost everywhere on $E = \cup_{i=1}^{n}[\alpha_i, \beta_i]$ is *D-integrable on $E$* if $f$ is $D$-integrable on each $[\alpha_i, \beta_i]$ and the value of integral is $(D)\int_E f = \sum_{i=1}^{n}(D)\int_{\alpha_i}^{\beta_i} f$. In the case of a complex-valued function $f$, it is *D-integrable on $E$* if both Re$f$ and Im$f$ are $D$-integrable with integrals $A$ and $B$, respectively, and the $D$-integral of $f$ is $A + iB$.

**Definition 3.9.** Let $E \in \mathcal{R}_{\mathcal{P}}$ be the image of $T \in \mathcal{R}_G$ under the mapping $\varphi$ (see (7)). Then a complex-valued function $f$ defined almost everywhere on $T$ is *D-integrable on $T$*, with the value $A$, if the function $F(x) = f\big(\varphi^{-1}(x)\big)$ is $D$-integrable on $E$ with the value $A$ in the sense of Definition 3.8, and we write $(D)\int_T f d\mu_G = A$.

**Remark 3.2.** It is easy to check out that the $D$-integral (in the sense of the above definition) is invariant under translation given by some "integer part" $[g]$ of some element $g \in G$.

We need the following theorem (see [9]).

**Theorem 3.1.** *Suppose that the sequence* (1) *is bounded and a function $f$ defined on $[a, b]$ is both $H_{\mathcal{P}}$-integrable and $D$-integrable on $[a, b]$ . If $\Phi$ is the indefinite*

*$H_{\mathcal{P}}$-integral of $f$ and the upper and lower $\mathcal{P}$-path derivatives of* $\mathrm{Re}\Phi$ *and* $\mathrm{Im}\Phi$ *are bounded at each point of* $[a,b]$*, except a countable set, then the values of* $H_{\mathcal{P}}$*- and* $D$*-integrals of* $f$ *on* $[a,b]$ *coincide.*

The next result is of great importance for our further consideration. It is a direct consequence of the above theorem, related to the real line setting, and of Definitions 3.5 and 3.9.

**Theorem 3.2.** *Suppose that the sequence* (1) *is bounded and* $E \in \mathcal{R}_{\mathcal{P}}$ *is the image of* $T \in \mathcal{R}_G$ *under the mapping* $\varphi$ *(see(7)). If a* $H_G$*-integrable on* $T \in R_G$ *function* $f$ *and its indefinite* $H_G$*-integral* $\Phi$ *are such that* $f\big(\varphi^{-1}(x)\big)$ *and* $\Phi\big(\varphi^{-1}(x)\big)$ *satisfy the conditions of the previous theorem, then the values of* $H_G$*- and* $D$*-integrals of* $f$ *on* $T$ *coincide.*

## 4. An inversion formula for integral transforms convergent to a $D$-integrable function

Using the notations for the cosets of $X_0$ and $G_0$ introduced in Section 2, we denote $[\chi]\backslash_{G_0}$ by $h_s$ if $[\chi] \in X_0^s$. In the same way, if $[g] \in G_0^k$, we denote $[g]\backslash_{X_0}$ by $h'_k$. It can be easily checked that the sequence $\{h_s\}_{s=0}^{\infty}$ includes all the characters of $G_0$ and $\{h'_k\}_{k=0}^{\infty}$ includes all the characters of $X_0$. This sequence $\{h_s\}$ having been translated on $[0,1]$ using the mapping (7), forms an orthonormal system $\{\xi_s\}$ on $[0,1]$, where $\xi_s(x) = h_s\big(\varphi^{-1}(x)\big)$ (see [1]).

The following theorem is proved in [7].

**Theorem 4.1.** *Suppose that the sequence* (1) *is bounded, the series* $\sum a_s\xi_s$ *with respect to the above system* $\{\xi_s\}$ *converges a.e. on* $[0,1]$ *to a function* $f$ *and everywhere on* $[0,1] \setminus Q_{\mathcal{P}}$*, we have*

$$\limsup_{n\to\infty} |S_n(x)| < +\infty, \tag{10}$$

*where* $S_n = \sum_{s=0}^{n-1} a_s\xi_s$*. Then* $f$ *is* $H_{\mathcal{P}}$*-integrable on* $[0,1]$ *(in the sense of Definition* 3.4 *and* $\sum a_s\xi_s$ *is* $H_{\mathcal{P}}$*-Fourier series of* $f$*.*

We can reformulate this result in terms of the system $\{h_s\}$ defined on a group.

**Theorem 4.2.** *Let* $G_0$ *be a compact abelian zero-dimensional group such that the sequence* (1) *is bounded and let* $\{h_s\}_s$ *be a system defined as above. If the series* $\sum_{s=0}^{\infty} a_s h_s$ *converges a.e. on* $G_0$ *to a function* $f$ *and everywhere on* $G_0\backslash\varphi^{-1}(Q_{\mathcal{P}})$ *we have*

$$\limsup_{n\to\infty} |\sigma_n(g)| < +\infty, \tag{11}$$

*where* $\sigma_n = \sum_{s=0}^{n-1} a_s h_s$*, then* $f$ *is* $H_{G_0}$*-integrable on* $G_0$ *and* $\sum a_s h_s$ *is the Fourier series of* $f$ *in the sense of integral defined in Definition* 3.5*.*

It is known (see [7, Lemma 3.2]) that the condition (10) implies the boundedness of upper and lower derivatives of $\mathrm{Re}\Phi$ and $\mathrm{Im}\Phi$, where $\Phi$ is an indefinite $H_{\mathcal{P}}$-integral of $f$.

Combining this with the last two theorems and with Theorem 3.2, we get.

**Theorem 4.3.** *Let $G_0$ and the system $\{h_s\}_s$ be as in Theorem 4.2 If the series $\sum_{s=0}^{\infty} a_s h_s$ converges a.e. on $G_0$ to a D-integrable function $f$ and everywhere on $G_0 \setminus \varphi^{-1}(Q_{\mathcal{P}})$ the inequality (11) holds, then $\sum a_s h_s$ is the Denjoy-Fourier series of $f$ (in the sense of the integral defined in Definition 3.9.*

This theorem is a generalization of a similar result related to some special case of zero-dimensional group (see [5, Theorem 2]).

We also need the following theorem, which can be established using the same arguments as in [8, Theorem 8]:

**Theorem 4.4.** *The partial sums $\sigma_{m_n}(f,g)$ of the $H_G$-Fourier series (with respect to the system of characters of $G_0$) of a $H_G$-integrable on $G_0$ function $f$ are convergent to $f$ almost everywhere on $G_0$.*

A continuum analog of the series $\sum_{s=0}^{\infty} a_s h_s$ is the integral transform

$$\int_X a(\chi)(g,\chi)\,\mathrm{d}\mu_X$$

with appropriately defined improper integral on $X$. So, the next theorem, which gives an inversion formula for this transform, can be considered as a generalization of Theorem 4.3.

**Theorem 4.5.** *Assume that $G$ is a group described in Section 2 such that the sequence (1) is bounded, $X$ is its dual group. Let $a(\chi)$ be a locally $H_X$-integrable function and*

$$\lim_{s\to\infty} (H_X) \int_{\cup_{i=0}^{s} X_0^i} a(\chi)(g,\chi)\,\mathrm{d}\mu_X = f(g) \tag{12}$$

*a.e. on $G$, where $f$ is a D-integrable function on each $T \in \mathcal{R}_G$. Moreover, let us have everywhere on $G \setminus \varphi^{-1}(Q_{\mathcal{P}})$*

$$\limsup_{s\to\infty} \left| (H_X) \int_{\cup_{i=0}^{s} X_0^i} a(\chi)(g,\chi)\,\mathrm{d}\mu_X \right| < +\infty. \tag{13}$$

*Then, the function $a(x)$ can be recovered from $f$ by the following inversion formula:*

$$a(\chi) = \lim_{n\to\infty} (D) \int_{G_{-n}} f(g)\overline{(g,\chi)}\,\mathrm{d}\mu_G \qquad \text{a.e. on} \quad X.$$

P r o o f. Suppose that $g \in G_0^k$. Then, for each $s = 1, 2, \dots$, we have, according to (6),

$$\begin{aligned}
&(H_X) \int\limits_{\cup_{i=0}^s X_0^i} a(\chi)(g, \chi)\,\mathrm{d}\mu_X \\
&= (H_X) \int\limits_{\cup_{i=0}^s X_0^i} a(\chi)\big(\{g\}, [\chi]\big) \cdot \big([g], [\chi]\big) \cdot \big([g], \{\chi\}\big)\,\mathrm{d}\mu_X \\
&= \sum_{i=0}^{s} (H_X) \int\limits_{X_0^i} a(\chi)\big(\{g\}, [\chi]\big) \cdot \big([g], [\chi]\big) \cdot \big([g], \{\chi\}\big)\,\mathrm{d}\mu_X \qquad (14)\\
&= \sum_{i=0}^{s} h_i\big(\{g\}\big)(H_X) \int\limits_{X_0^i} a(\chi)\big([g], [\chi]\big) h_k'\big(\{\chi\}\big)\,\mathrm{d}\mu_X.
\end{aligned}$$

Now, for any $g \in G_0^k$ for which the limit (12) exists, we get:

$$f(g) = \lim_{s\to\infty} \sum_{i=0}^{s} h_i\big(\{g\}\big)(H_X) \int\limits_{X_0^i} a(\chi)\big([g], [\chi]\big) h_k'\big(\{\chi\}\big)\,\mathrm{d}\mu_X. \qquad (15)$$

So, for such $g \in G_0^k$, the function $f(g)$ is the sum of series with respect to the system $\{h_i\}$ with coefficients

$$b_i = (H_X) \int\limits_{X_0^i} a(\chi)\big([g], [\chi]\big) h_k'\big(\{\chi\}\big)\,\mathrm{d}\mu_X$$

and this series is convergent almost everywhere on $G_0^k$.

In the same way, (13) and (15) imply, for any $g \in G_0^k \setminus \varphi^{-1}(Q_{\mathcal{P}})$

$$\limsup_{s\to\infty} \left| \sum_{i=0}^{s} h_i\big(\{g\}\big)(H_X) \int\limits_{X_0^i} a(\chi)\big([g], [\chi]\big) h_k'\big(\{\chi\}\big)\,\mathrm{d}\mu_X \right| < \infty. \qquad (16)$$

Then, by Theorem 4.3, the coefficients $b_i$ are the Denjoy-Fourier coefficients of the $D$-integrable function $p(t) = f\big([g] + t\big)$ on $G_0$, i.e.,

$$\begin{aligned}
b_i = (H_X) \int\limits_{X_0^i} a(\chi)\big([g], [\chi]\big) h_k'\big(\{\chi\}\big)\,\mathrm{d}\mu_X &= (D) \int\limits_{G_0} p(t)\overline{h_i(t)}\,\mathrm{d}\mu_G \\
&= (D) \int\limits_{G_0^k} f(g)\overline{h_i\big(\{g\}\big)}\,\mathrm{d}\mu_G
\end{aligned} \qquad (17)$$

(the last equality is justified by Remark 3.2). By property 3) of Section 2, $\big([g],[\chi]\big)$ is constant when $g \in G_0^k$ and $\chi \in X_0^i$. Hence, (17) implies

$$(H_X)\int\limits_{X_0^i} a(\chi)h_k'\big(\{\chi\}\big)\,\mathrm{d}\mu_X = (D)\int\limits_{G_0^k} f(g)\overline{\big([g],[\chi]\big)h_i\big(\{g\}\big)}\,\mathrm{d}\mu_G. \tag{18}$$

The rest of the proof follows the lines of the proof of Theorem 9 of [8].

For a fixed $i$, the value

$$(H_X)\int\limits_{X_0^i} a(\chi)h_k'\big(\{\chi\}\big)\,\mathrm{d}\mu_X$$

is the Fourier coefficient in the system $\{\overline{h_k'}\}$ of the $H_X$-integrable function $a(\chi) = a\big([\chi]+\{\chi\}\big)$ considered as a function of $\{\chi\} \in X_0$. Therefore, Theorem 4.4 being applied to the above system and the appropriate partial sums, implies

$$\sum_{k=0}^{m_{-n}}(H_X)\int\limits_{X_0^i} a(\chi)h_k'\big(\{\chi\}\big)\,\mathrm{d}\mu_X\cdot\overline{h_k'\big(\{\chi\}\big)} \to a(\chi) \qquad \text{a. e. on} \quad X_0^i.$$

Then, using (18) and (6), we get

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{k=0}^{m_{-n}}(H_X)\int\limits_{X_0^i} a(\chi)h_k'\big(\{\chi\}\big)\,\mathrm{d}\mu_X\cdot\overline{h_k'\big(\{\chi\}\big)}\\
&= \lim_{n\to\infty}\sum_{k=0}^{m_{-n}}(D)\int\limits_{G_0^k} f(g)\overline{\big([g],[\chi]\big)h_i\big(\{g\}\big)}\,\mathrm{d}\mu_G\cdot\overline{h_k'\big(\{\chi\}\big)}\\
&= \lim_{n\to\infty}\sum_{k=0}^{m_{-n}}(D)\int\limits_{G_0^k} f(g)\overline{h_i\big(\{g\}\big)h_k'\big(\{\chi\}\big)\big([g],[\chi]\big)}\,\mathrm{d}\mu_G\\
&= \lim_{n\to\infty}(D)\int\limits_{\cup_{k=0}^{m_{-n}}G_0^k} f(g)\overline{\big(\{g\},[\chi]\big)\big([g],\{\chi\}\big)\big([g],[\chi]\big)}\,\mathrm{d}\mu_G\\
&= \lim_{n\to\infty}(D)\int\limits_{G_{-n}} f(g)\overline{(g,\chi)}\,\mathrm{d}\mu_G = a(\chi) \qquad \text{a. e. on} \quad X_0^i.
\end{aligned}$$

The last equality is true for any $i$. Therefore, we can write

$$\lim_{n\to\infty}(D)\int\limits_{G_{-n}} f(g)\overline{(g,\chi)}\,\mathrm{d}\mu_G = a(\chi) \qquad \text{a. e. on} \quad X.$$

This completes the proof. □

**Remark 4.1.** As both $D$-integral and $H_X$-integral obviously include the Lebesgue integral, the last theorem can be formulated, in particular, for functions $a$ and $f$ being locally Lebesgue integrable on the groups $X$ and $G$, respectively.

## REFERENCES


[1] AGAEV, G. N.—VILENKIN, N. YA.—DZHAFARLI, G. M.—RUBISTEIN, A. I.: *Multiplicative System of Functions and Harmonic Analysis on Zero-Dimensional Groups.* "Ehlm", Baku, 1981. (In Russian)

[2] BONGIORNO, B.—DI PIAZZA, L.—SKVORTSOV, V. A.: *The Ward property for a $\mathcal{P}$-adic basis and the $\mathcal{P}$-adic integral*, J. Math. Anal. Appl. **285** (2003) 578–592.

[3] GOLUBOV, B.—EFIMOV, A.—SKVORTSOV, V.: *Walsh Series and Transforms. Theory and Applications.* Math. Appl. (Soviet Ser.), Vol. 64, Kluwer Academic Publishers, Dordrecht, 1991.

[4] GORDON, R. A.: *The Integrals of Lebesgue, Denjoy, Perron, and Henstock Theory*, Grad. Stud. Math., Vol. 4, Amer. Math. Soc., Providence, RI, 1994.

[5] SKVORTSOV, V. A.—KOROLEVA, M. P.: *Series in multiplicative systems convergent to Denjoy-integrable functions*, Sb. Math. **186** (1995), No. 12, 1821–1842; translation from Mat. Sb. **186** (1995), No. 12, 129–150.

[6] SKVORTSOV, V. A.—TULONE, F.: *The $\mathcal{P}$-adic Henstock integral in the problem of function representations by multiplicative transforms,* Vestnik Moskov. Univ. Ser. I Mat. Mekh. **2005** (2005), No. 3, 41–44, translation in Moscow Univ. Math. Bull. **60** (2005), No. 3, 19–22.

[7] SKVORTSOV, V. A.—TULONE, F.: *On the problem of recovering the coefficients of series with respect to characters of zero-dimensional groups*, Tatra Mt. Math. Publ. **34** (2006), 307–320.

[8] SKVORTSOV, V. A.—TULONE, F.: *Henstock type integral in harmonic analysis on zero-dimensional groups*, J. Math. Anal. Appl. **322** (2006), 621–628.

[9] SKVORTSOV, V. A.—TULONE, F.: *Integration of both the derivatives with respect to $\mathcal{P}$-paths and approximative derivatives*, Mat. Zametki **85** (2009), No. 2, 283–291, translation in Math. Notes **85** (2009), 260–266.

[10] RUDIN, W.: *Fourier Analysis on Groups.* John Wiley & Sons, New York, 1967.


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